Lecture 15:

Orthogonality

Definition: Let V be an inner product space. We say $\vec{x}, \vec{y} \in V$ are <u>orthogonal</u> (or perpendicular) if $\langle \vec{x}, \vec{y} \rangle = 0$.

A subset $S \subset V$ is called orthogonal if any two distinct vectors in S are orthogonal.

A <u>unit vector</u> in V is a vector $\vec{x} \in V$ with $\|\vec{x}\| = 1$.

A subset $S \subset V$ is called <u>orthonormal</u> if S is orthogonal and all vectors in S are unit vectors.

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e.g. Let H be the space of continuous complex-valued functions on $[0, 2\pi]$. We have inner product defined by:

$$\langle f, g \rangle \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt \quad \text{for } f, g \in H$$

For any $n \in \mathbb{Z}$ ← integer, let (i ← $\sqrt{-1}$)

$$f_n(t) = e^{int} \overset{\text{def}}{:=} \cos nt + i \sin nt \quad \text{for } t \in [0, 2\pi]$$

and consider $S = \{ f_n : n \in \mathbb{Z} \} \subset H$

For any $m \neq n$, we have:

$$\langle f_m, f_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{imt}\, \underset{\underset{e^{-int}}{\shortparallel}}{\overline{e^{int}}}\, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt = \frac{1}{2\pi} \left(\frac{1}{i(m-n)} \right) e^{i(m-n)t} \Big|_0^{2\pi} = 0$$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos (m-n)t\, dt + i \frac{1}{2\pi} \int_0^{2\pi} \sin (m-n)t\, dt \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt$$

Also, $\langle f_n, f_n \rangle = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{int}\, \underset{\substack{\shortparallel \\ e^{-int}}}{\overline{e^{int}}}\, dt = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} 1\, dt = 1$

$\therefore$ S is orthonormal subset of H.

Definition: Let V be an inner product space. A subset of V is an **orthonormal basis** for V if it is an ordered basis which is orthonormal.

Proposition: Let V be an inner product space and $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthogonal subset of V consisting of non-zero vectors.

Then: $\forall\, \vec{y} \in \mathrm{Span}(S)$,

$$\vec{y} = \sum_{i=1}^{k} \left(\frac{\langle \vec{y}, \vec{v}_i \rangle}{\| \vec{v}_i \|^2} \right) \vec{v}_i$$

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Proof: Write $\vec{y} = \sum_{i=1}^{k} a_i \vec{v}_i$ for some $a_1, a_2, \ldots, a_k \in F$.

Take inner product with $\vec{v}_j$ on both sides gives:

$$\langle \vec{y}, \vec{v}_j \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}_i, \vec{v}_j \rangle = a_j \|\vec{v}_j\|^2$$

Corollary 1: If, in addition to above, S is orthonormal, then $\forall \vec{y} \in \mathrm{Span}(S)$,

$$\vec{y} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

Corollary 2: Let S be an orthogonal subset of an inner product space V consisting of non-zero vectors. Then, S is linearly independent.

Proof: If $\overbrace{\sum_{i=1}^{k} a_i \vec{v}_i}^{\in \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_k\})} = \vec{0}$ for some $\vec{v}_1, \dots, \vec{v}_k \in S$ and $a_1, a_2, \dots, a_k \in F$,

By previous proposition,

$$a_i = \cancelto{0}{\langle \vec{0}, \vec{v}_i \rangle} \Big/ \|\vec{v}_i\|^2 = 0 \quad \text{for } i = 1, 2, \dots, k. \quad //$$

<u>Prop</u>: Let V be an inner product space and $S_n = \{\vec{w}_1, \dots, \vec{w}_n\}$ be a linearly independent subset of V. Define:

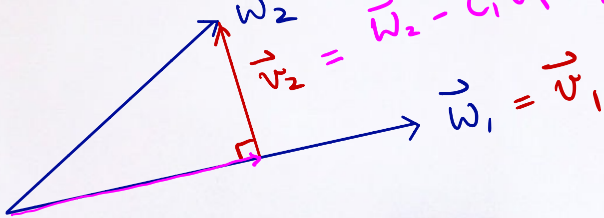
$$S_n' = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \text{ where } \vec{v}_1 = \vec{w}_1 \text{ and}$$

for $k = 2, \dots, n$,

$$\vec{v}_k \overset{\text{def}}{=} \vec{w}_k - \sum_{j=1}^{k-1} \left(\frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \right) \vec{v}_j$$

Then: S_n' is orthogonal and $\text{Span}(S_n') = \text{span}(S_n)$

$$\vec{v}_2 = \vec{w}_2 - c_1 \vec{v}_1 \quad \text{where } c_1 = \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}$$



Proof: We prove by induction on n.

For $n=1$, we simply have $S_1' = S_1$. $\therefore$ The statement is obviously true.

Suppose the statement is true for $n = m-1$.

That's, $S_{m-1}' = \{\vec{v}_1, \dots, \vec{v}_{m-1}\}$ is orthogonal and

$$\text{Span}(S_{m-1}') = \text{Span}(S_{m-1})$$

Induction hypothesis

Now, consider a lin. independent subset $S_m = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_{m-1}, \vec{w}_m\}$

Then: for $\vec{v}_m \overset{\text{def}}{=} \vec{w}_m - \sum_{j=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \vec{v}_j$, we have:

$$\langle \vec{v}_m, \vec{v}_i \rangle = \langle \vec{w}_m, \vec{v}_i \rangle - \sum_{j=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \langle \vec{v}_j, \vec{v}_i \rangle \quad \text{for } i = 1, \dots, m-1$$

$$= \cancel{\langle \vec{w}_m, \vec{v}_i \rangle} - \cancel{\frac{\langle \vec{w}_m, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}} \cancel{\langle \vec{v}_i, \vec{v}_i \rangle} = 0$$

$\therefore$ S_m' is orthogonal.

Also, $\vec{v}_m \neq \vec{0}$ since otherwise, $\vec{w}_m \in \text{Span}(S_{m-1}')$

$$\begin{aligned} &= \text{Span}(S_{m-1}) \\ &= \text{Span}(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{m-1}\}) \end{aligned}$$

contradicting the condition that S_m is linearly independent.

Hence, $S_m' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m-1}, \vec{v}_m\}$ is orthogonal subset consisting of non-zero vectors. $\therefore$ S_m' is linearly independent.

Also, $\underbrace{\text{Span}(S_m')}_{\dim = m} \subset \underbrace{\text{Span}(S_m)}_{\dim = m} \Rightarrow \text{Span}(S_m') = \text{Span}(S_m)$

The above construction of an orthogonal basis is called <u>Gram-Schmidt process.</u>